Perron Numbers and Lind's Theorem

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Master's Thesis

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To Jes, Jer, Ry, Al, Ma, and Pa

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A special thanks to Lee Mosher for all his help and support, and also for $Edit_1 \ldots Edit_n$ (where n is some large prime number)

Abstract.

Thurston's last paper *Entropy in Dimension One* contains a theorem that connects Perron Numbers, $\mathrm{Out}(F_n)$, and Train Track Maps. In trying to solve a problem about constructing Train Track maps from self-maps of trees, we quickly saw that the problem reduced down to looking at Lind's theorem. Thus, in this paper we discuss preliminary definitions, consequences of Lind's theorem, study a specific example for the 2×2 case, and finally explore Thurston's proof of Lind's theorem.

Introduction.

Before William P. Thurston passed away[1], one of the last papers he wrote was *Entropy in Dimension One* [6], and the paper discusses a method for constructing train track maps given any expansion constant which is a Perron number. To put it simply, train track maps are homotopy equivalences between finite connected graphs which do not "back track" along their paths. Backtracking, more rigorously, refers to cancellations along each single edge for which the map iterates over. The expansion constant is the largest positive eigenvalue of the matrix corresponding to the train track map, and Perron numbers are positive real algebraic integers that are strictly greater than their Galois conjugates in absolute value.

Nevertheless, we will mainly focus on the proof of Doug Lind's Theorem as reproved by Thurston in [6] (which is a type of converse of to the Perron-Frobenius Theorem):

Theorem.(Perron-Frobenius [1]) Let A be a non-negative, non-zero, irreducible, real square matrix. Then

(1) A has a positive, real, right eigenvector, unique up to multiplication by a positive real number; the associated eigenvalue $\mathrm{PF}(A)$ is positive;
(2) $\mathrm{PF}(A) = \mathrm{PF}(A^T)$;
(3) for every eigenvalue τ of the matrix A, it holds $|\tau| \leq \mathrm{PF}(A)$.

Definition.([1]) Given a square matrix A, we say A is *reducible* if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

where X, Z are square matrices. If a matrix A is not reducible, then A is said to be *irreducible*.

Lemma.([1]) Let A be a non-negative non-zero irreducible matrix of size $n \times n$. Then the matrix $B = \sum_{i=0}^{n-1} A^i$ is positive.

Definition.([2]) Given a square matrix A, we say A is *aperiodic* if $A^n > 0$ for some positive integer n .

[1]I had the fortune of hearing Thurston speak for the last time at the June 2012 Cornell Topology Festival.

Definition. Given a square matrix A, the *spectral radius* of A is the largest absolute value of its eigenvalues.

In particular, if A has integer coefficients then its eigenvalues are algebraic integers. This is clearly seen from expanding the characteristic determinant and obtaining the characteristic polynomial - the polynomial is always monic with integer coefficients. All the entries along the diagonal multiply with one another only once leaving the highest power term with a coefficient of 1.

Theorem 1. (Lind's Theorem [3]) If λ is a Perron number, then there is an irreducible and aperiodic (equivalently, the matrix has a positive power) non-negative integral matrix with spectral radius λ.

We will look at several specific examples of Lind's Theorem which address some questions related to that theorem. For instance, let n be the least dimension of an $n \times n$ matrix with PF eigenvalue λ, and let $k = \deg(\lambda)$. Is n bounded by k (how big does the matrix get)? For small n, is there a $1-1$ correspondence between n and k? Is n bounded for small k at least? For degree 2 Perron numbers the answer is in the affirmative that the dimension of the matrix is bounded. More specifically, we can always find a 2×2 ergodic (aperiodic) non-negative integral matrix for a given degree 2 Perron number.

However, this is where it stops as Lind points out an example of a degree 3 Perron number that needs at least a 4×4 matrix. In fact, he constructs a 10×10 matrix which has this particular λ as the dominant eigenvalue. The question of, "How big can the matrix be?" is answered in an email from Lind which confirms even just for the degree 3 case the corresponding matrix can be arbitrarily large.

We will also go over the machinery of Lind's proof as engineered by Thurston. The basic idea is to first pick a Perron Number λ and find its ring of integers $\mathcal{O}_\lambda$. Next, find a generating set (spanning set) for the lattice formed by the ring of integers under some embedding, and look at the bottom of the cone formed by the spanning vectors. The geometry of the hull of the cone constructed from interior integer points will determine the dimension of the matrix which has spectral radius λ. So, let's drop a λ "coin" into the Lind machine and see how it becomes a matrix.

Perron Numbers.

A Perron number λ is a positive real algebraic integer whose Galois conjugates are all strictly less than λ in absolute value. An algebraic integer is a complex number which is the root of a monic polynomial whose coefficients are over $\mathbb{Z}$.

For example, $\lambda = 1 + \sqrt{2}$ is a Perron number of degree 2 because its minimal polynomial is $x^2 - 2x - 1$ which is monic over $\mathbb{Z}$, next for $\lambda's$ Galois conjugate in absolute value, $|1-\sqrt{2}| < 1+\sqrt{2}$. However, $\lambda = \sqrt{2}$ is not a strict Perron number because for its conjugate we have $|-\sqrt{2}| \leq \sqrt{2}$. We call $\sqrt{2}$ a Weak Perron number which is a Perron number whose Galois conjugates are less than or equal to it in absolute value. In addition, more examples of Perron numbers include:

λ	polynomial
1.057050...	$x^{13} - x - 1$
1.052710...	$x^{14} - x - 1$
1.047595...	$\frac{x^{17}-x^4-1}{x^2-x+1}$*
1.048984...	$x^{15} - x - 1$
1.045751...	$x^{16} - x - 1$.

*This rational function is actually equal to the polynomial

$$x^{15} + x^{14} - x^{12} - x^{11} + x^9 + x^8 - x^6 - x^5 + x^3 - x - 1.$$

In [6], it says "Lind conjectured that the smallest Perron number of degree $d \geq 2$ should have minimal polynomial $x^d - x - 1$." Although, this turns out to be false as the above table of examples shows for $d = 15$. This was first shown to be false by Boyd who has calculated all of the smallest Perron numbers of degree $d \leq 12$. Here is the new conjecture ([7]):

Conjecture (Lind - Boyd). The smallest Perron number of degree $d \geq 2$ has minimal polynomial

$$\begin{aligned} x^d - x - 1 \quad &\text{if} \quad d \not\equiv 3, 5 \mod 6, \\ \frac{x^{d+2} - x^4 - 1}{x^2 - x + 1} \quad &\text{if} \quad d \equiv 3 \mod 6, \\ \frac{x^{d+2} - x^2 - 1}{x^2 - x + 1} \quad &\text{if} \quad d \equiv 5 \mod 6. \end{aligned}$$

This conjecture has not been resolved yet.

Examples

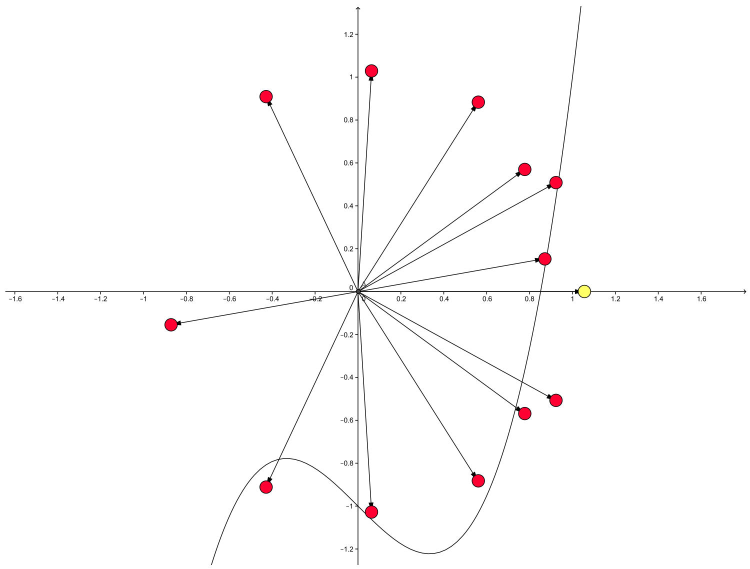


Figure 1: The roots of $x^{13} - x - 1$ in the complex plane with its one real root root denoted by the yellow dot.

When someone who knows some math first reads Lind's Theorem the initial question that comes to mind is " Is the size of the matrix equal to the degree of the Perron number? If not, is the dimension bounded at least, and what is the minimal dimension for the matrix?" Let's look at the degree 2 case - can we always find a 2×2 matrix given a degree 2 Perron number? The answer is yes and will be proved below.

Nevertheless, going up only more degree to 3 the answer is "No." In [3] Lind points out an example of a degree 3 Perron number that needs at least a 4×4 matrix. Take $p(x) = x^3 + 3x^2 - 15x - 46$; this polynomial has dominant eigenvalue $\lambda \approx 3.8916$, and Galois conjugates $\lambda_2 \approx -3.2142, \lambda_3 \approx -3.6775$. The general characteristic polynomial for a 3×3 matrix is given by $x^3 - \text{Tr}(A)x^2 + bx - \det(A)$ where b is a constant that depends on $\text{Tr}(A)$ and $\text{Tr}(A^2)$. The important thing to note is the trace is -3 for a 3×3 matrix with characteristic polynomial $p(x)$, but since the matrix we are looking for is non-negative the $\text{Tr}(A) \geq 0$, and so there cannot be a 3×3 matrix for $p(x)$. In fact, Lind constructs a 10×10 matrix with characteristic polynomial $p(x)$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 9 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}.$$

The dominant eigenvalue is still indeed λ as the characteristic polynomial factors as

$$(x+1)(x^3+3x^2-15x-46)(x^6-4x^5-4x^4-6x^2-50x+24)$$

and the roots of the degree 6 (irreducible) term are approximately 3.2042, 0.5134, -1.8277 $\pm$ 0.1641i, 1.9689 $\pm$0.6751i. Is this the minimally sized non-negative integer matrix with dominant eigenvalue λ? If not, where exactly does the size of the matrix lie between 4 and 10? The question is unresolved, and furthermore finding the answer to that question is a non-trivial piece of research. However, what Lind does know is that the corresponding matrix can arbitrarily large for a given Perron Number of degree greater than 2.

The Ring of Integers $\mathcal{O}_K$

What proceeds is a brief discussion of a geometric representation of algebraic numbers summarized from [5] - embedding a number field K in a real vector space. More specifically, given a number field K there is an associated subring called the ring of integers denoted $\mathcal{O}_K$. The ring of integers of a field K is the collection of elements of K that are solutions to some monic polynomial with integer coefficients, and in this case the ring of integers $\mathcal{O}_K$ is realized as a lattice in $\mathbb{C}^s \oplus \mathbb{R}^t$. A number field K is a finite degree extension of $\mathbb{Q}$ and has the form $K = \mathbb{Q}(\lambda)$ for some $\lambda \in \mathbb{R}$. Before defining the embedding, let's describe $\mathbb{C}^s \oplus \mathbb{R}^t$: it is the set of all $(s+t)$-tuples of the form

$$v = (v_1, ..., v_s; v_{s+1}, ..., v_{s+t})$$

where the $v_1, ..., v_s \in \mathbb{C}$ and the $v_{s+1}, ..., v_{s+t} \in \mathbb{R}$. In fact, it is a vector space over $\mathbb{R}$ with dimension $2s+t$. Defining coordinate-wise addition and multiplication also gives $\mathbb{C}^s \oplus \mathbb{R}^t$ a ring structure.

Now let's define the embedding:

$$F : \mathbb{Q}(\lambda) \hookrightarrow \mathbb{C}^s \oplus \mathbb{R}^t$$

$$F(\alpha) = (F_1(\alpha), ..., F_s(\alpha); F_{s+1}(\alpha), ..., F_{s+t}(\alpha)) \text{ for } \alpha \in \mathbb{Q}(\lambda)$$

where the F_i are all the injective homomorphisms $\mathbb{Q}(\lambda) \to \mathbb{C}$ minus the corresponding complex conjugates $\bar{F}_i$. F is a ring homomorphism since each of its components is an injective homomorphism. When we restrict ourselves to the ring of integers $\mathcal{O}_K$, the image of $F|_{\mathcal{O}_K}\colon \mathcal{O}_K \hookrightarrow \mathbb{C}^s \oplus \mathbb{R}^t$ is a lattice of dimension $2s+t$.

Here is a corollary from Marcus's *Number Fields* [4] which will be very useful since we will working exclusively in a real quadratic field for the following examples.

Corollary 2. Let m be a square-free integer. The set of algebraic integers in the quadratic field $\mathbb{Q}[\sqrt{m}]$ is

(1) $\left\{a + b\sqrt{m} : a, b \in \mathbb{Z}\right\}$; if $m \equiv 2$ or $3 \pmod 4$

OR

(2) $\left\{\dfrac{a + b\sqrt{m}}{2} : a, b \in \mathbb{Z}, a \equiv b \pmod 2\right\}$; if $m \equiv 1 \pmod 4$.

As mentioned before, the ring of integers $\mathcal{O}_K$ of a number field K is the collection of elements of K which are solutions to some monic polynomial with integer coefficients. The simplest example is to take $K = \mathbb{Q}$. We can look at the monic polynomial $x^2 - 3 = 0$ which has solutions $x = \pm\sqrt{3}$ neither of which is an element in $\mathbb{Q}$. However, the solutions of $x^2 - 4 = 0$ are in $\mathbb{Q}$. It turns out that $\mathcal{O}_K = \mathbb{Z}$ and can be generated by the solutions to $x - n = 0$ for some $n \in \mathbb{Z}$.

As a more complicated example take $\lambda = \sqrt{7}$ and adjoin to $\mathbb{Q}$ to get $K = \mathbb{Q}(\sqrt{7})$. Is $\frac{1+\sqrt{7}}{3} \in \mathcal{O}_\lambda$? No, as it is the root of the non-monic irreducible polynomial $3x^2 - 2x - 2 = 0$. By Corollary 2 in [4] and the fact that $7 \equiv 3 \pmod 4$, the ring of integers $\mathcal{O}_\lambda = \mathbb{Z}[\sqrt{7}]$ (where each element is a solution to the polynomial $x^2 - 2ax + a^2 - 7b^2$ for some $a, b \in \mathbb{Z}$).

Example. Let $\lambda = \frac{21+3\sqrt{5}}{2}$. Firstly, λ is indeed a Perron number of degree 2 as its minimal polynomial is given by $p(x) = x^2 - 21x + 99$ and its Galois conjugate is $\frac{21-3\sqrt{5}}{2}$ which is smaller than λ. Let's construct a 2×2 matrix with λ as the dominant eigenvalue. Here, $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{5})$ and the ring of integers $\mathcal{O}_\lambda$ is given by

$$\left\{ \frac{a + b\sqrt{5}}{2} : a \equiv b \bmod 2; a, b \in \mathbb{Z} \right\}.$$

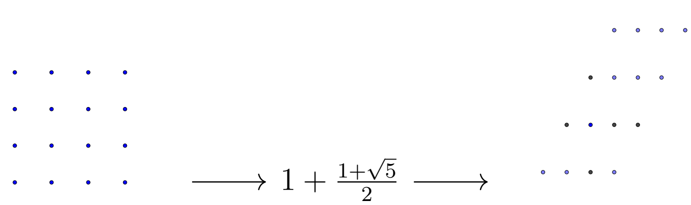
A basis can be given by $\left\{ m \cdot \frac{1}{2} + n \cdot \frac{\sqrt{5}}{2} | m \equiv n \bmod 2 \right\}$. However, this precludes many pairs of integers; we want this to be over all of $\mathbb{Z} \oplus \mathbb{Z}$. So let $n = m + 2k$, plug in and rename the variables to get

$$\frac{(2m+n) + n\sqrt{5}}{2} = m + n\frac{1+\sqrt{5}}{2}.$$

So we see that we can take a basis for $\mathcal{O}_\lambda$ as $\left\{ 1, \frac{1+\sqrt{5}}{2} \right\}$. Let's also write down the isomorphism and the embedding

$$(m, n) \mapsto m \cdot 1 + n \cdot \frac{1+\sqrt{5}}{2} \mapsto \left(m \cdot 1 + n \cdot \frac{1+\sqrt{5}}{2}, m \cdot 1 + n \cdot \frac{1-\sqrt{5}}{2} \right)$$

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathcal{O}_\lambda \longrightarrow f(\mathcal{O}_\lambda)$$



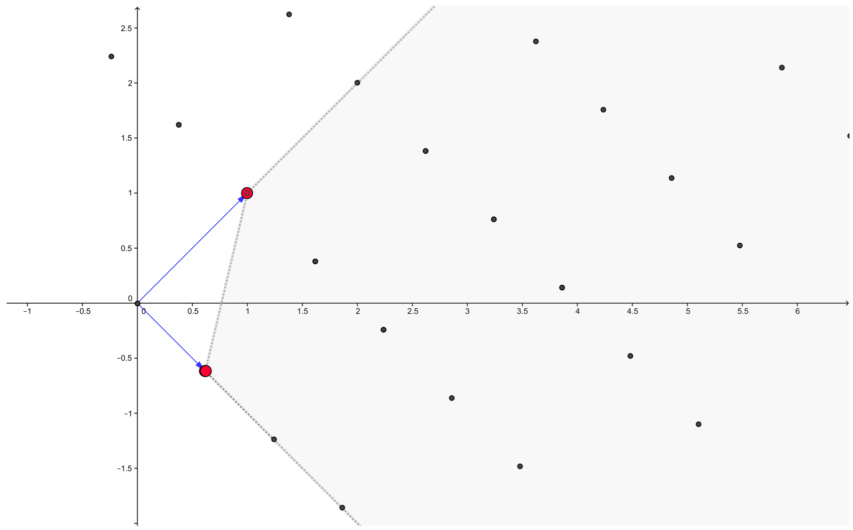


Figure 2: Lattice cone and hull for $\lambda = \frac{21+3\sqrt{5}}{2}$, and $\mathcal{O}_\lambda$ basis $\left\{1, \frac{1+\sqrt{5}}{2}\right\}$.

where the lattice is given by $f(\mathcal{O}_\lambda)$. The basis vectors for the lattice are given by

$$\left\{f(1), f\left(\frac{1+\sqrt{5}}{2}\right)\right\} = \left\{(1,1), \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)\right\}.$$

Multiply the basis elements by λ to get a new basis, and now write down the matrix:

$$1 \mapsto \frac{21+3\sqrt{5}}{2} = 9 \cdot 1 + 3 \cdot \frac{1+\sqrt{5}}{2}$$
$$\frac{1+\sqrt{5}}{2} \mapsto \frac{36+24\sqrt{5}}{4} = 3 \cdot 1 + 12 \cdot \frac{1+\sqrt{5}}{2}$$

$$\begin{pmatrix} 9 & 3 \\ 3 & 12 \end{pmatrix}.$$

$$
\begin{array}{ccc}
\mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{\begin{pmatrix}9 & 3\\ 3 & 12\end{pmatrix}} & \mathbb{Z}\oplus\mathbb{Z} \\
\begin{pmatrix}1 & 0\\ 0 & \frac{1+\sqrt{5}}{2}\end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix}1 & 0\\ 0 & \frac{1+\sqrt{5}}{2}\end{pmatrix} \\
\mathcal{O}_\lambda & & \mathcal{O}_\lambda \\
\begin{pmatrix}1 & 1\\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix}1 & 1\\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix} \\
\mathbb{R}^2 & \xrightarrow{\begin{pmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{pmatrix}} & \mathbb{R}^2
\end{array}
$$

Now we see maybe there is always a 1st quadrant and 4th quadrant basis for degree 2 case.

Lemma 1. Let λ be a Perron number of degree 2. Given an embedding f of the ring of integers $\mathcal{O}_\lambda \hookrightarrow \mathbb{R}^2$, there exists a $\{1\text{st quadrant}, 4\text{th quadrant}\}$ basis for the lattice $f(\mathcal{O}_\lambda)$. Moreover, the cone determined by the basis is invariant under the action of λ.

Proof. Let λ be a Perron number of degree 2. By the above Corollary 2, we have two cases for the ring of intgers $\mathcal{O}_\lambda$: elements of this ring are either written as $a + b\sqrt{d}$ or as $\frac{a+b\sqrt{d}}{2}$ depending on the value of d (reminding d is square-free).

Case 1. $d \equiv 2, 3 \mod 4$ with basis $\{1, \sqrt{d}\}$.

Define $f : \mathcal{O}_\lambda \hookrightarrow \mathbb{R}^2$

$$f(m + n\sqrt{d}) = (m + n\sqrt{d}, m - n\sqrt{d}).$$

We will show that $\{f(1), f(\sqrt{d})\} = \{(1, 1), (\sqrt{d}, -\sqrt{d})\}$ is a basis for the lattice $f(\mathcal{O}_\lambda)$.

First note that $(1, 1)$ is in the first quadrant and $(\sqrt{d}, -\sqrt{d})$ is in the 4th quadrant. The following map[2] E_d is a $\mathbb{Z}-$module isomorphism since it is a homomorphism and bijective.

[2]Notice we are using the notation E_d for the general embedding F instead of F_d, as not to confuse the reader with the monomorphisms F_i.

$$(m,n) \xmapsto{E_d} (m+n\sqrt{d}, m-n\sqrt{d})$$

$$\begin{aligned}E_d((m_1,n_1)+(m_2,n_2)) &= E_d(m_1+n_1,m_2+n_2)\\ &= ((m_1+n_1)+(m_2+n_2)\sqrt{d},(m_1+m_2)-(n_1+n_2)\sqrt{d})\\ &= (m_1+n_1\sqrt{d}+m_2+n_2\sqrt{d},m_1-n_1\sqrt{d}+m_2-n_2\sqrt{d})\\ &= E_d(m_1,n_1)+E_d(m_2,n_2)\end{aligned}$$

$1-1$: Let $v_i = (m_i, n_i)$

If $E_d(v_1) = E_d(v_2)$

$$\begin{aligned}&\Rightarrow (m_1+n_1\sqrt{d},m_1-n_1\sqrt{d}) = (m_2+n_2\sqrt{d},m_2-n_2\sqrt{d})\\ &\Rightarrow m_1+n_1\sqrt{d} = m_2+n_2\sqrt{d}\\ &\Rightarrow m_1=m_2, n_1=n_2.\end{aligned}$$

Surjective: Yes, since E_d maps onto its image.

Finally, the image of a basis for $\mathbb{Z}\oplus\mathbb{Z}$ is a basis for $E_d(\mathbb{Z}\oplus\mathbb{Z})$.

Invariance of the cone under T_λ (case 1):

We have to show $T_\lambda(C(\mu_1,\mu_2)) \subset C(\mu_1,\mu_2)$ where $\mu_1 = \langle 1,1\rangle$, $\mu_2 = \langle\sqrt{d},-\sqrt{d}\rangle$, and

$$T_\lambda = \begin{pmatrix} m+n\sqrt{d} & 0 \\ 0 & m-n\sqrt{d}\end{pmatrix}.$$

Note that $n, m \geq 0$ since we are working in the 1st and 4th quadrants. Moreover, $n \neq 0$ since $\deg(\lambda) = 2$. Let $v \in T_\lambda(C)$ which means

$$\begin{aligned}v &= T_\lambda(a_1\mu_1+a_2\mu_2)\\ &= a_1T_\lambda(\mu_1)+a_2T_\lambda(\mu_2).\end{aligned}$$

Then show

$$v = b_1\mu_1+b_2\mu_2 \text{ for some } b_1,b_2\in\mathbb{Z} \text{ (which implies } v\in C).$$

$$T_\lambda(\mu_1) = \begin{pmatrix} m+n\sqrt{d} & 0 \\ 0 & m-n\sqrt{d} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{d} \\ -\sqrt{d} \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} \sqrt{d} \\ -\sqrt{d} \end{pmatrix}.$$

Pick $b_1 = m$ and $b_2 = n$. Similarly,

$$T_\lambda(\mu_2) = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix} \begin{pmatrix} \sqrt{d} \\ -\sqrt{d} \end{pmatrix} = \begin{pmatrix} d \\ d \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} \sqrt{d} \\ -\sqrt{d} \end{pmatrix}.$$

Pick $b_1 = nd$ and $b_2 = m$. When we put these solutions into a matrix

$$\begin{pmatrix} m & nd \\ n & m \end{pmatrix},$$

we obtain a matrix with the same eigenvalues ($\sqrt{d}$ and $-\sqrt{d}$).

Case 2. $d \equiv 1 \bmod 4$ with basis $\left\{1, \frac{1+\sqrt{d}}{2}\right\}$.
Define $f : \mathcal{O}_\lambda \hookrightarrow \mathbb{R}^2$

$$f\left(m + n\frac{1+\sqrt{d}}{2}\right) = \left(m + n\frac{1+\sqrt{d}}{2}, m + n\frac{1-\sqrt{d}}{2}\right).$$

We will show that $\left\{f(1), f\left(\frac{1+\sqrt{d}}{2}\right)\right\} = \left\{(1,1), \left(\frac{1+\sqrt{d}}{2}, \frac{1-\sqrt{d}}{2}\right)\right\}$ is a basis for the lattice $f(\mathcal{O}_\lambda)$. Again note that $(1,1)$ is in the first quadrant and $\left(\frac{1+\sqrt{d}}{2}, \frac{1-\sqrt{d}}{2}\right)$ is in the fourth quadrant since

$$\begin{aligned} & d > 2 \\ \Rightarrow & \sqrt{d} > \sqrt{2} > 1 \\ \Rightarrow & 1 - \sqrt{d} < 0. \end{aligned}$$

Invariance of the cone under T_λ (case 2):

We have to show $T_\lambda(C(\mu_1, \mu_2)) \subset C(\mu_1, \mu_2)$ where $\mu_1 = \langle 1, 1 \rangle$, $\mu_2 = \left\langle \frac{1+\sqrt{d}}{2}, \frac{1-\sqrt{d}}{2} \right\rangle$, and

$$T_\lambda = \begin{pmatrix} m + n\frac{1+\sqrt{d}}{2} & 0 \\ 0 & m + n\frac{1-\sqrt{d}}{2} \end{pmatrix}.$$

Let $v \in T_\lambda(C)$ which means

$$\begin{aligned} v &= T_\lambda(a_1\mu_1 + a_2\mu_2) \\ &= a_1 T_\lambda(\mu_1) + a_2 T_\lambda(\mu_2). \end{aligned}$$

Then show

$$v = b_1\mu_1 + b_2\mu_2 \text{ for some } b_1, b_2 \in \mathbb{Z} \text{ (which implies } v \in C).$$

$$T_\lambda(\mu_1) = \begin{pmatrix} \frac{1+\sqrt{d}}{2} & 0 \\ 0 & \frac{1-\sqrt{d}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{d}}{2} \\ \frac{1-\sqrt{d}}{2} \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} \frac{1+\sqrt{d}}{2} \\ \frac{1-\sqrt{d}}{2} \end{pmatrix}.$$

Pick $b_1 = 0$ and $b_2 = 1$. Similarly,

$$T_\lambda(\mu_2) = \begin{pmatrix} \frac{1+\sqrt{d}}{2} & 0 \\ 0 & \frac{1-\sqrt{d}}{2} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{d}}{2} \\ \frac{1-\sqrt{d}}{2} \end{pmatrix} = \begin{pmatrix} \left(\frac{1+\sqrt{d}}{2}\right)^2 \\ \left(\frac{1-\sqrt{d}}{2}\right)^2 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} \frac{1+\sqrt{d}}{2} \\ \frac{1-\sqrt{d}}{2} \end{pmatrix}.$$

Pick[3] $b_1 = n\frac{d-1}{4}$ and $b_2 = m + n$. When we put these solutions into a matrix

$$\begin{pmatrix} m & n\frac{d-1}{4} \\ n & m+n \end{pmatrix},$$

Note that in checking the invariance of the cone, we have constructed a matrix with the same eigenvalues.

The map E_d is also a $\mathbb{Z}-$module isomorphism since it is a homomorphism and bijective:

$$(m, n) \xmapsto{E_d} \left(m + n\tfrac{1+\sqrt{d}}{2}, m + n\tfrac{1-\sqrt{d}}{2}\right).$$

Illustration of Method: without a basis.

A basis for $\mathbb{R}^2$ is $\{e_1, e_2\} = \{\langle 1, 1\rangle, \langle 2, 1\rangle\}$. Construct the matrix

[3]Expand the square and use the good ol' $+1 - 1$. Also remember for this case $d \equiv 1 \mod 4$ so b_1 is an integer.

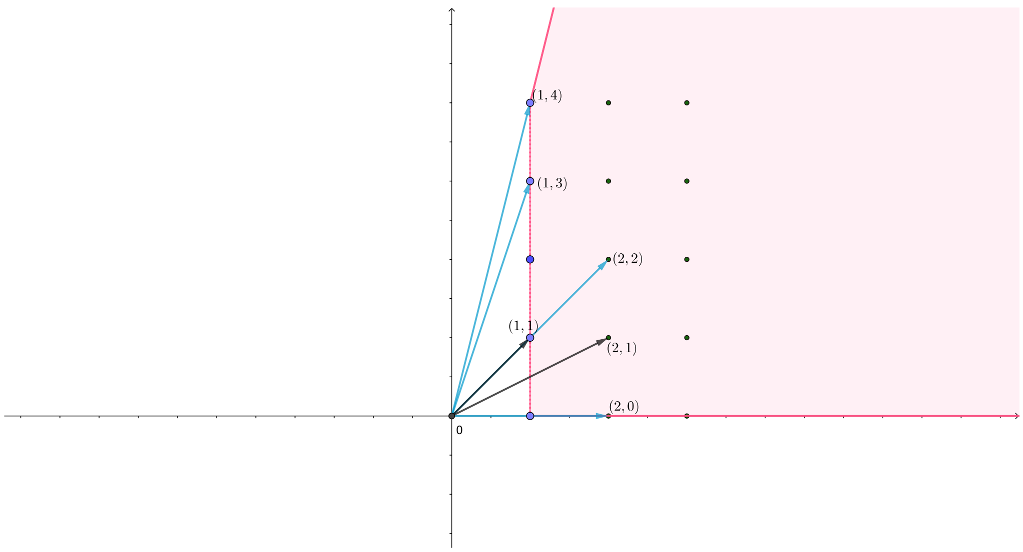


Figure 3: A spanning set of four vectors for $\mathbb{R}^2$ to illustrate the method of computing the matrix when the spanning set is not a basis.

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

where the columns of the matrix are the two basis vectors. Note the four blue vectors $(1,4), (1,3), (2,2)$, and $(2,0)$ in Figure 3 generate the lattice $\mathbb{Z} \oplus \mathbb{Z}$. Then extracting the five points

$$e_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, e_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, e_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, e_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

from the hull of the cone we obtain five equations:

$$Me_i = a_{i1}e_1 + a_{i2}e_2 + a_{i3}e_3 + a_{i4}e_4 + a_{i5}e_5.$$

For Me_1, we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}.$$

After eliminating one of the variables (a_4), choose the values for the other four: $a_1, a_2, a_3,$ and a_5. Since this is an underdetermined system a solution is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 4 \end{pmatrix}.$$

Iterating this procedure four more times we end up with a 5×5 matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 4 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Notice that M and A have exactly the same eigenvalues, namely $\frac{1}{2}(3+\sqrt{5})$ and $\frac{1}{2}(3-\sqrt{5})$.

$$\begin{aligned} E_d((m_1, n_1) + (m_2, n_2)) &= E_d(m_1 + n_1, m_2 + n_2) \\ &= \left((m_1 + n_1) + (m_2 + n_2)\frac{1+\sqrt{d}}{2}, (m_1 + m_2) + (n_1 + n_2)\frac{1-\sqrt{d}}{2} \right) \\ &= \left(m_1 + n_1\frac{1+\sqrt{d}}{2} + m_2 + n_2\frac{1-\sqrt{d}}{2}, \right. \\ &\qquad \left. m_1 + n_1\frac{1+\sqrt{d}}{2} + m_2 + n_2\frac{1-\sqrt{d}}{2} \right) \\ &= E_d(m_1, n_1) + E_d(m_2, n_2) \end{aligned}$$

$1-1$: Let $v_i = (m_i, n_i)$

If $E_d(v_1) = E_d(v_2)$

$$\Rightarrow \left(m_1 + n_1\frac{1+\sqrt{d}}{2}, m_1 + n_1\frac{1-\sqrt{d}}{2}\right) = \left(m_2 + n_2\frac{1+\sqrt{d}}{2}, m_2 + n_2\frac{1-\sqrt{d}}{2}\right)$$
$$\Rightarrow m_1 + n_1\frac{1+\sqrt{d}}{2} = m_2 + n_2\frac{1+\sqrt{d}}{2}$$
$$\Rightarrow m_1 = m_2, n_1 = n_2.$$

Surjective: Yes, since E_d maps onto its image. Finally, the image of a basis for $\mathbb{Z} \oplus \mathbb{Z}$ is a basis for $E_d(\mathbb{Z} \oplus \mathbb{Z})$.

□

Proposition. Let λ be a degree 2 Perron number with minimal monic polynomial $p(x)$. Then there exists a nonnegative 2×2 integer matrix M with characteristic polynomial $p(x)$. Also M is irreducible.

Proof. The proof of the invariance of the cone (determined by the 1st, 4th quadrant basis) in Lemma 1 begets two nonnegative integer matrices,

$$\begin{pmatrix} m & nd \\ n & m \end{pmatrix} \text{ and } \begin{pmatrix} m & n\frac{d-1}{4} \\ n & m+n \end{pmatrix},$$

each of which have the desired polynomial $p(x)$ for the corresponding cases.

□

This proposition implies for a Perron number λ of degree 2, we can always find a nonnegative 2×2 integer matrix for which λ is the PF eigenvalue. It is also noted that the matrix is not unique as we can have a different matrix with the same eigenvalue by picking a different basis, such as $\left\{1, \frac{1+3\sqrt{5}}{2}\right\}$, which gives the matrix:

$$\begin{pmatrix} 10 & 1 \\ 11 & 11 \end{pmatrix}.$$

If, instead of a basis, we just have a spanning set (which contains more elements than the basis), can we write down a higher dimensional matrix for the same eigenvalue? Can the original, smaller matrix be "embedded" inside a larger matrix[4]? Well, here is a 3×3 matrix with PF eigenvalue $\lambda = \frac{21+3\sqrt{5}}{2}$:

$$\begin{pmatrix} 10 & 1 & 2 \\ 9 & 11 & 5 \\ 0 & 1 & 9 \end{pmatrix}.$$

I guessed this matrix (trial and error) using WolframAlpha. A good question is "Given the matrix, is there a way to work backwards to the two basis vectors which determine the lattice?"

Speaking of interesting matrices, here is a 6×6 matrix with $\lambda = 4$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 2 & 0 \\ 6 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

Degree 3

So far we have looked at degree 2 examples for the ring of integers. Now let's examine two degree 3 examples where $K = \mathbb{Q}(\sqrt[3]{d})$.

Example. Let $\lambda = 4 - 7\sqrt[3]{10} + 20\left(\frac{1+\sqrt[3]{10}+\sqrt[3]{10^2}}{3}\right)$, which has minimal polynomial $p(x) = x^3 - 32x^2 + 408x - 31554$. This means we will be working in the number field $\mathbb{Q}(\sqrt[3]{10})$. In [4] page 38, we find a basis for the ring of integers $\mathcal{O}_\lambda \subset \mathbb{Q}(\sqrt[3]{d})$ as being one of the two sets:

$$\left\{1, \sqrt[3]{d}, \sqrt[3]{d^2} : d \not\equiv \pm 1 \pmod{9}\right\}$$

$$\left\{1, \sqrt[3]{d}, \frac{1 \pm \sqrt[3]{d} + \sqrt[3]{d^2}}{3} : d \equiv \pm 1 \pmod{9}\right\}.$$

[4]Notice the the upper left 2×2 block of the 3×3 matrix is almost the prior 2×2 matrix. Although, maybe it is just a coincidence.

Here d must be cube-free and square-free. Also the $+/-$ signs correspond in the obvious way. For an embedding E_{10} we just need to choose a map that is $1-1$ and a homomorphism.

$$E_{10} : (m, n, q) \mapsto \left(m + n\sqrt[3]{10} + q\sqrt[3]{10^2}, m + n(-4.4 + 27.2\mathrm{i}) + q(-732.1 + 243.8\mathrm{i})\right)$$

In general, when we have one real root λ_1 and two complex roots λ_2, λ_3 (complex conjugate pairs, $\lambda_3 = \bar{\lambda_2}$) we have the embedding:

$$(m, n, q) \mapsto m + n\lambda_1 + q\lambda_1^2 \mapsto$$

$$(m + n\lambda_1 + q\lambda_1^2, m + n\lambda_2 + q\lambda_2^2) \approx$$

$$(m + n\lambda_1 + q\lambda_1^2, m + n\mathrm{Re}[\lambda_2] + q\mathrm{Re}[\lambda_2^2], n\mathrm{Im}[\lambda_2] + q\mathrm{Im}[\lambda_2^2]).$$

Then we have to compute $\{f(1), f(\lambda), f(\lambda^2)\}$(the real and imaginary parts were already taken into account):

$$\begin{aligned} f(1) &= (1, 1, 0) \\ f(\lambda) &= \left(\sqrt[3]{10}, \frac{-\sqrt[3]{10}}{2}, \frac{\sqrt[3]{10}\sqrt{3}}{3}\right) \\ f(\lambda^2) &= (2.599, -.799, .718). \end{aligned}$$

From [4] also,"If d is not square-free, let k denote the product of all primes which divide d twice (so $m = hk^2$ with h and k square-free and relatively prime); then an integral basis for $\mathcal{O}_\lambda$ consists of "

$$\left\{1, \sqrt[3]{d}, \frac{\sqrt[3]{d}}{k} \quad \text{if} \quad d \not\equiv \pm 1 \pmod 9\right\}$$
$$\left\{1, \sqrt[3]{d}, \frac{k^2 \pm k^2\sqrt[3]{d} + \sqrt[3]{d}^2}{3k} \quad \text{if} \quad d \equiv \pm 1 \pmod 9\right\}.$$

Example. Let $\lambda = \frac{1+\sqrt[3]{10}+\sqrt[3]{10^2}}{3}$ which has minimal polynomial $p(x) = x^3 - x^2 - 3x - 3$.

Use the embedding E_d from above.

$$f(1) = (1, 1 + 0i) \approx (1, 1, 0)$$
$$f(\lambda) = (\lambda, \lambda_2) \approx (\lambda, \mathrm{Re}[\lambda_2], \mathrm{Im}[\lambda_2]) = (2.5987, -0.7993, 0.7179)$$
$$f(\lambda^2) = (\lambda^2, \lambda_2^2) \approx (\lambda^2, \mathrm{Re}[\lambda_2^2], \mathrm{Im}[\lambda_2^2] = (6.7532, 0.6388, 0.5153)$$

These are the basis vectors for the lattice.

$$1 \mapsto \lambda = 0 \cdot 1 + 1 \cdot \lambda + 0 \cdot \lambda^2 \to (0, 1, 0)$$

$$\lambda \mapsto 0 \cdot 1 + 0 \cdot \lambda + 1 \cdot \lambda^2 = (0, 0, 1)$$

$$\lambda^2 \mapsto \lambda^3 = 3 + 3\lambda + \lambda^2 \to (3, 3, 1).$$

This gives the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 3 & 1 \end{pmatrix}$$

It is worth noting that similar examples in the degree 3 case (and higher) may not be [strong] Perron because they are either not algebraic integers or the Galois conjugates are larger in absolute value then the real root being looked at.

Lattice Cones

Recall the definition of the embedding which takes a number field and suspends it in the space $\mathbb{C}^s \oplus \mathbb{R}^t$ as a lattice:

$$F : \mathbb{Q}(\lambda) \hookrightarrow \mathbb{C}^s \oplus \mathbb{R}^t$$

$$F(\alpha) = (F_1(\alpha), ..., F_s(\alpha); F_{s+1}(\alpha), ..., F_{s+t}(\alpha)) \text{ for } \alpha \in \mathbb{Q}(\alpha)$$

where the F_i are all the injective homomorphisms $\mathbb{Q}(\lambda) \to \mathbb{C}$ minus the corresponding complex conjugates $\bar{F}_i$.

Definition. Given a Perron number λ, the *Perron vector* is the corresponding eigenvector for the eigenvalue λ. See Figure 5.

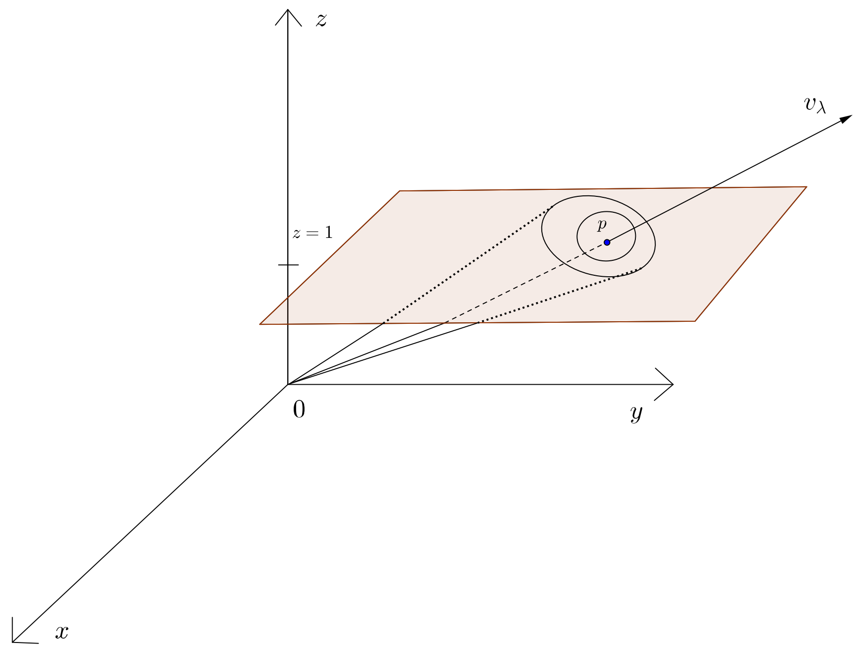


Figure 4: The Perron vector contained in the cone projected onto the $z = 1$ plane.

Definition. Let C be a cone, λ be a Perron number, and T_λ be a linear transformation. Then by λC we mean $T_\lambda C$.

Definition. A *lattice cone spanned by* $\mu_1, ..., \mu_k$ in a vector space $V \subset \mathbb{C}^s \oplus \mathbb{R}^t$ is a set C of the form

$C(\mu_1, ..., \mu_k) = \{a_1\mu_1 + \cdots + a_k\mu_k | a_i \geq 0; \mu_i \in \mathcal{O}\}$ where $a_1, ..., a_K$ are not all zero.

Definition. The *lattice convex hull spanned by* $\mu_1, ..., \mu_k$ of a lattice cone is the set (see Figure 6)

$$\mathcal{H}(\mu_1, ..., \mu_k) = C(\mu_1, ..., \mu_k) \cap (\mathcal{O} - \{0\}).$$

Proof of Lind's theorem by Thurston [6].
This will be done in three parts.

Part 1. Let λ be a Perron number. Then there exists a lattice cone P contained in C and containing λC ($\lambda P \subset \lambda C \subset P \subset C$).

Proof: Let λ be a Perron number. Take C to be the cone defined by all rays emanating from the origin, and piercing through the unit disk (D) in the $x_n = 1$ hyperplane. Define

$$\hat{f}(x_1, ..., x_{n-1}, x_n) = (\lambda_1 x_1, ..., \lambda_{n-1} x_{n-1}, \lambda x_n)$$

as our linear transformation. The λ_i are the Galois conjugates of λ, and the last coordinate slot, x_n, is the Perron slot. If $p = (a_1, ..., a_{n-1}, 1)$ (where $\sum_{i=1}^{n} a_i^2 \leq 1$) is a point inside the unit disk (see Figure 5), then $\hat{f}(p) = (\lambda_1 a_1, ..., \lambda_{n-1} a_{n-1}, \lambda)$ and $\frac{1}{\lambda}\hat{f}(p) = \left(\frac{\lambda_1}{\lambda} a_1, ..., \frac{\lambda_{n-1}}{\lambda} a_{n-1}, 1\right)$. We want to show $\frac{1}{\lambda}\hat{f}(x_1, ..., x_{n-1}, 1)$ is a contraction mapping (where x is a point inside the unit disk). We will use $d(\hat{f}(x), \hat{f}(y))^2 \leq k^2 d(x, y)^2$ (where $k < 1$) since it implies $d(\hat{f}(x), \hat{f}(y)) \leq k d(x, y)$ and is easier to work with.

$$\begin{aligned}
\left\|\frac{1}{\lambda}\hat{f}(x) - \frac{1}{\lambda}\hat{f}(y)\right\|^2 &= \left\|\left(\frac{\lambda_1}{\lambda}x_1, ..., \frac{\lambda_{n-1}}{\lambda}x_{n-1}, 1\right) - \left(\frac{\lambda_1}{\lambda}y_1, ..., \frac{\lambda_{n-1}}{\lambda}y_{n-1}, 1\right)\right\|^2 \\
&= \left\|\left(\frac{\lambda_1}{\lambda}(x_1 - y_1), ..., \frac{\lambda_{n-1}}{\lambda}(x_{n-1} - y_{n-1}), 0\right)\right\|^2 \\
&= \left(\frac{\lambda_1}{\lambda}\right)^2 |x_1 - y_1|^2 + \cdots + \left(\frac{\lambda_{n-1}}{\lambda}\right)^2 |x_{n-1} - y_{n-1}|^2 \\
&\leq \max_i \left\{\left(\frac{\lambda_i}{\lambda}\right)^2\right\} \left[|x_1 - y_1|^2 + \cdots + |x_{n-1} - y_{n-1}|^2\right].
\end{aligned}$$

Take $k = \max_i \left\{\frac{\lambda_i}{\lambda}\right\} < 1$ because $|\lambda| > |\lambda_i|$ for all $i = 1, 2, ..., (n-1)$ since λ is a Perron number.

Now, the set of elements in $\mathbb{Q}(\lambda)$ with final coordinate $x_n = 1$ are dense in the $x_n = 1$ hyperplane. This is a result because $\mathbb{Q}(\lambda)$ is dense in $\mathbb{C}^s \oplus \mathbb{R}^t$. This can been see as follows: set $n = 2s + t$; there also exists a vector space isomorphism

$$\mathbb{C}^s \oplus \mathbb{R}^t \longrightarrow \mathbb{R}^n = \mathbb{R}^{2s} \oplus \mathbb{R}^t,$$

which restricts to an abelian group isomorphism

$$\mathcal{O}_\lambda \longrightarrow \mathbb{Z}^n = \mathbb{Z}^{2s} \oplus \mathbb{Z}^t,$$

and which restricts to an abelian group isomorphism

$$\mathbb{Q}(\lambda) \longrightarrow \mathbb{Q}^n = \mathbb{Q}^{2s} \oplus \mathbb{Q}^t.$$

Therefore, we obtain[5] the sequence

$$\underbrace{\mathbb{Q}(\lambda) \xrightarrow{\text{dense}} \mathbb{Q}^n \xrightarrow{\text{dense}} \mathbb{R}^n \xrightarrow{\text{dense}} \mathbb{C}^s \oplus \mathbb{R}^t}_{\text{dense}}.$$

Since this holds true, we can always find rational points within the interior of the unit disk (vectors emanating from the origin with rational coordinates)

[5]Note that we have used the density of $\mathbb{Q} \subset \mathbb{R}$

to construct a polyhedron[6] in the $x_n = 1$ hyperplane. Let P be a polyhedron with vertex set V - we need two conditions on V. One condition is each point of V comes sufficiently close to some point of ∂C, and the other is each point of ∂C comes sufficiently close to some point of V. The meaning[7] of "sufficiently close" is specified by some $\delta > 0$: each point of V should within some δ of some point of ∂C, and vice-versa (see Figure 7).

Then there exists a $\delta > 0$ so that if V is chosen to satisfy these two conditions then ∂P satisfies the following set inclusions (in particular the first). Let $\epsilon = \min\{d(C, \frac{1}{\lambda}\hat{f}(C))\}$:

$$\begin{aligned}
&\partial P \subset N_{\frac{\epsilon}{3}}(\partial C)\\
&\text{and}\\
&\hat{f}(\partial P) \subset N_{\frac{\epsilon}{3}}(\hat{f}(\partial C))\\
&\Rightarrow \hat{f}(P) \subset \text{int}(P).
\end{aligned}$$

□

Part 2. The lattice convex hull $\mathcal{H}(\mu_1, ..., \mu_k)$ is finitely generated.

Proof: A topological proof is given in [6] with a footnote (10) by John Milnor correcting the open sets not forming a basis (see Figure 6 for a possible scenario). Thurston also mentions this a standard fact: see [2] p.286 where the word "semigroup" is mentioned twice.

□

Part 3. Matrix Construction.

Proof: Combining parts 1 and 2, for each $i = 1, ..., k$ one can solve the equation

$$\hat{f}(\mu_i) = a_{i1}\mu_1 + \cdots + a_{ik}\mu_k$$

for non-negative integer values of the a_{ij}, and the desired matrix is simply $[a_{ij}]$ (recall the Illustration of the Method on page 12).

□

[6]The polyhedron need not be inscribed, even if the pictures depict that.

[7]There is a phrase for this relation - ∂C and V are *delta Hausdorff close.*

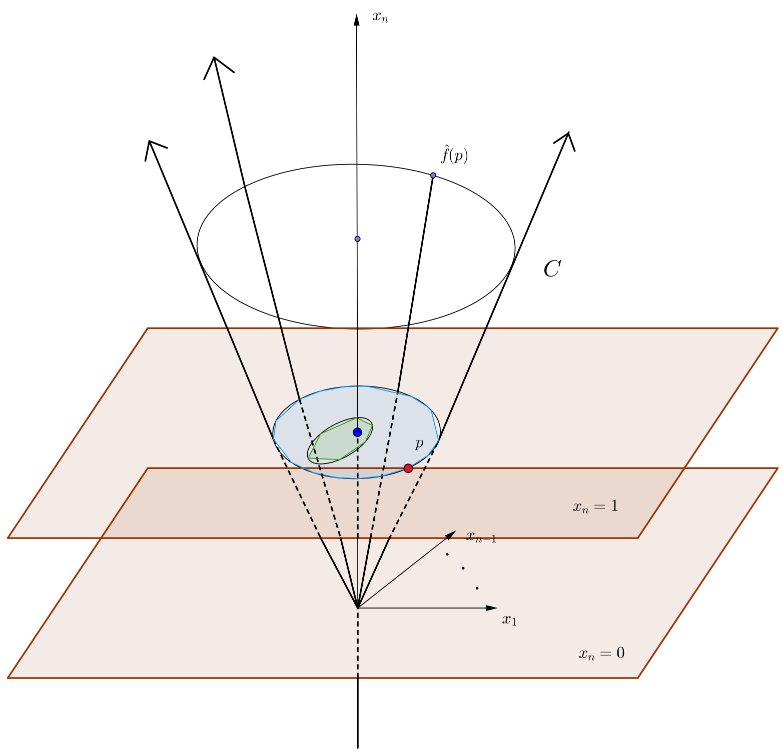


Figure 5: All points in the cone C get mapped back into the cone by the contraction mapping $\hat{f}$.

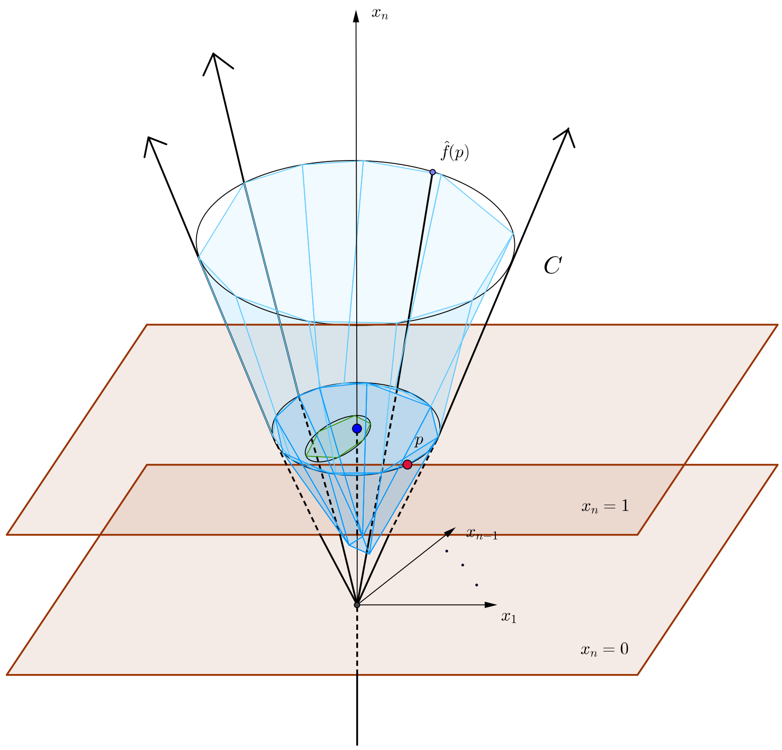


Figure 6: The polyhedral cone P constructed from rational points.

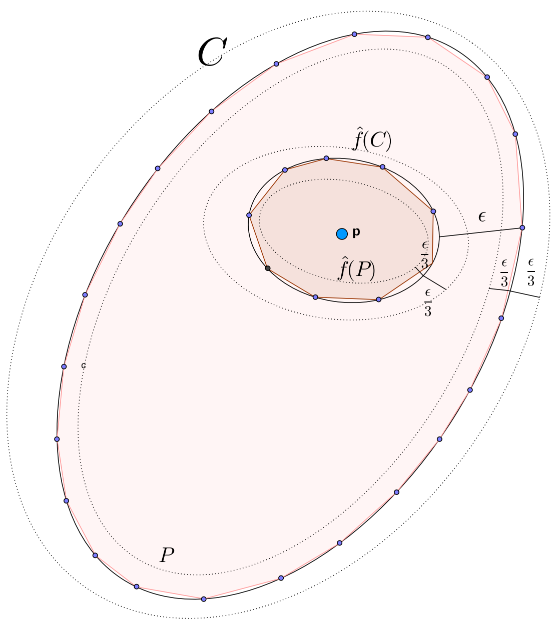


Figure 7: This picture shows that image of the polygon P mapped under $\hat{f}$ does not seep outside the cone. It should also be noted that while the picture depicts the polygon inscribed in the cone, this need not be the case.

References.

[1] Bogopolski, oleg. *Introduction to Group Theory*. European Mathematical Society. 2008.

[2] Lind, Doug. *The entropies of topological Markov shifts and a related class of algebraic integers*. Ergodic Theory Dynam. Systems, 4(2):283-300, 1984.

[3] Lind, Doug. *Entropies and Factorizations of Topological Markov Shifts*. Bulletin of the Mathematical Society. Volume 9, Number 2, September 1983.

[4] Marcus, Daniel. *Number Fields*. Springer. 1977.

[5] Tall,D.O. and Stewart, I.N. *Algebraic Number Theory*. Chapter 8. Chapman and Hall Ltd. 1979.

[6] Thurston, William P. *Entropy In Dimension One*. Proceedings of the Banff conference on Frontiers in Complex Dynamics.2014.

[7] Wu, Qiang. *The Smallest Perron Numbers*. MATHEMATICS OF COMPUTATION. Volume 79, Number 272, October 2010, Pages 23872394. April 26, 2010.